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OPTIMAL CONTROL LAWS FOR A PENSION PLAN WITH AND WITHOUT RETURN CLAUSE AND VOLATILITY RISK

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ABSTRACT

This paper merged together the study of optimal control laws for a pension plan with and without return clause under Heston volatility model. An investment model comprising of members' monthly contributions, return accumulations with risk free interest to dead members' families for the case with return clause and investment in one risk free asset and two risky assets is presented. Since the mean variance utility function is time inconsistent, the game theoretic approach is used to establish an optimization problem from the extended Hamilton Jacobi Bellman (HJB) equation. Furthermore, the optimal control laws for the three assets and the efficient frontier are obtained using variable separation method by solving the extended HJB equations. Finally, Numerical simulations were presented to demonstrate the effects of some parameters on the optimal control laws with observations that the optimal control law for risk free asset decreases continuously with time while that of the risky assets increases continuously with time.

Keywords: Extended HJB equation, pension plan, optimal control laws, volatility risk, return of premium.

INTRODUCTION

In finance, the management of portfolios is a critical issue as it concerns investment in assets which are modelled by Brownian motions. Due to some degree of randomness in modelling the market prices of these assets, the study of optimal control laws as it governs investments in financial markets has drawn so much attention from researchers all over the world and also financial managers from different financial institutions such as insurance companies, commercial bank, pension fund system etc have equally engaged in the study. According to (Antolin *et al*., 2010) the importance of pension fund system in preparing for members retirement cannot be over emphasized since it gives members the opportunity to plan for their old age. Currently, there are two types of pension plan namely; the defined benefit (DB) plan (see Delong *et al*., 2008; Chen and Hao, 2013; Josa-Fombellida, 2012) and the defined contribution pension plan (DC) (see Devolder *et al*., 2003; Gao, 2008; Akpanibah and Oghenoro, 2018; Akpanibah and Osu, 2018). These authors studied the optimal investment problems under different assumptions in both DB and DC pension funds.

presently, there have been an increase in the study of optimal control laws governing investments when pension fund managers are mandated to refund the contributions of members who lost their life during the accumulation phase since members of the scheme are faced with mortality risk; as a result of this (He and Liang, 2013) studied the optimal control laws for DC pension fund with return of premium clause; they assumed the returned fund is without interest and the remaining accumulations was equally shared among the surviving members. They considered investment in one risk free asset and one risky asset such that the risky asset was modelled by geometric Brownian motion. Sheng and Rong (2014) studied the optimal control laws with return clause where they considered investment in one risk free asset and a risky asset (stock) and assumed the stock market price is modelled by Heston volatility model. Osu *et al*. (2018) studied optimization problem with return of premium in a DC pension with multiple contributors; in their work, the stock market price was driven by constant elasticity of variance model (CEV) model. Li *et al*. (2017) studied equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under (CEV) model; they considered investments in treasury, stock and bond. In a recent study Akpanibah *et al*. (2020) investigated the optimal control law for a DC pension plan when the returned contributions are with predetermined interest; they considered investment in a risk free and a risky asset and assume the risky asset is modelled by Heston volatility model.

From the available literatures and to the best of our knowledge, no work have been done on optimal control laws for a pension plan ''with'' and ''without'' return

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clause that considers investment in a three assets such that the stock market price follows the Heston volatility model. Also the returned contributions are with risk free interest. This is the motivation behind this work.

1. The Investment Model

Consider a market which consist of onerisk-free asset and two risky assets; namely stock and loan. Let$(\Omega, \mathcal{F}, \mathbb{p})$ be a complete probability space such that Ω is a real space and $\mathbb{p}$ a probability measure satisfying the condition $0 \leq t \leq T$. $\{\mathcal{W}_s(t), \mathcal{W}_z(t), \mathcal{W}_l(t), \mathcal{W}_m(t) : t \geq 0\}$ are standard Brownian motions. $\mathcal{F}$is the filtration and denotes the information generated by the Brownian motions. Also, let the financial market be a complete and frictionless type and is continuously open over a given time interval $0 \leq t \leq T$, such that$T$is the retirement age of pension members.

Let $R_t(t)$ denote the price of the risk free asset and the price process is driven by

$$d\mathcal{R}_t(t) = r_1 \mathcal{R}_t(t)dt, \mathcal{R}_t(0) = 1, \quad (1.1)$$

$\mathcal{S}_t(t)$denotes the price of the stock which is modelled by the Heston's stochastic volatility as follows

$$d\mathcal{S}_t(t) = \left(r_1 + \Delta_1 \mathcal{Z}_t(t)\right)\mathcal{S}_t(t)dt + \sqrt{\mathcal{Z}_t(t)}\mathcal{S}_t(t)d\mathcal{W}_s \mathcal{S}_t(0) = s_0 \quad (1.2)$$

$$d\mathcal{Z}_t(t) = \hbar(\nabla - \mathcal{Z}_t)dt + n_1\sqrt{\mathcal{Z}_t}d\mathcal{W}_z, \ \mathcal{Z}_t(0) = z_0 \quad (1.3)$$

$\mathcal{L}_t(t)$denotes the price of the loan and its price process is described as follows

$$d\mathcal{L}_t(t) = (r_1 + \Delta_2)\mathcal{L}_t(t)dt + n_2\mathcal{L}_t(t)d\mathcal{W}_l(t) + n_3\mathcal{L}_t(t)d\mathcal{W}_m(t) \quad (1.4)$$

Here r_1 is the predetermined interest rate of the risk free asset and $\hbar, \nabla, \Delta_1, \Delta_2, n_1, n_2, n_3$ are positive constants and the four Brownian motions are such that $d\mathcal{W}_s(t)d\mathcal{W}_z = \varepsilon, d\mathcal{W}_s(t)d\mathcal{W}_l = d\mathcal{W}_s(t)d\mathcal{W}_m = d\mathcal{W}_z(t)d\mathcal{W}_l = d\mathcal{W}_z(t)d\mathcal{W}_m = d\mathcal{W}_l(t)d\mathcal{W}_m = 0$, where ε is the correlation coefficient of $\mathcal{W}_s(t)$ and $\mathcal{W}_z$satisfying the condition$-1 \leq \varepsilon \leq 1$.

Considering the time interval $[t, t+i]$, the differential form associated with the fund size is given as:

$$\mathcal{U}(t+i) = \begin{pmatrix} \mathcal{U}(t)\left(\mu_1 \frac{\mathcal{R}_{t+i}(t)}{\mathcal{R}_t} + \mu_2 \frac{\mathcal{S}_{t+i}(t)}{\mathcal{S}_t} + \mu_3 \frac{\mathcal{L}_{t+i}(t)}{\mathcal{L}_t}\right) + \\ b(i) - tbni\mathcal{X}_{k_0+t} - \mu_1 \mathcal{U}(t)\frac{\mathcal{R}_{t+i}(t)}{\mathcal{R}_t} i\mathcal{X}_{k_0+t} \end{pmatrix} \frac{1}{1-i\mathcal{X}_{k_0+t}} \quad (1.5)$$

$$\mathcal{U}(t+i) = \begin{pmatrix} \mathcal{U}(t)\begin{pmatrix} 1 + (1-\mu_2-\mu_3)\left(\frac{\mathcal{R}_{t+i}(t)-\mathcal{R}_t}{\mathcal{R}_t}\right)(1-i\mathcal{X}_{k_0+t}) \\ +\mu_2\left(\frac{\mathcal{S}_{t+i}(t)-\mathcal{S}_t}{\mathcal{S}_t}\right) + \mu_3\left(\frac{\mathcal{L}_{t+i}(t)-\mathcal{L}_t}{\mathcal{L}_t}\right) \end{pmatrix} \\ +bi - tbni\mathcal{X}_{k_0+t} \\ -(1-\mu_2-\mu_3)n\mathcal{U}(t)i\mathcal{X}_{k_0+t} \end{pmatrix} \Big(1 + i\mathcal{X}k0+t1-i\mathcal{X}k0+t \quad (1.6)$$

Whereμ_1, μ_2, and $\mu_3$are the fractions of the members wealth to be invested in cash, stock and loan respectively such that $\mu_1 = 1 - \mu_2 - \mu_3$, b is the members' contributions received by the pension fund at any given time, k_0, the initial age of accumulation phase, T, the time frame of the accumulation period such that $k_0 + T$ is the end age. $i\mathcal{X}_{k_0+t}$is the mortality rate from time $t$to $t+i$, bt is the accumulated contributions at time$t$, $tbni\mathcal{X}_{k_0+t}$ and $\mu_1 n\mathcal{U}(t)\frac{\mathcal{R}_{t+i}(t)}{\mathcal{R}_t} i\mathcal{X}_{k_0+t}$ aretheaccumulated contributions and risk free interest paid to the death members' family such that if $n = 0$, there is no return of contribution and if $n = 1$, there is return of contribution.

Following (He and Liang, 2013), we have

$$i\mathcal{X}_{k_0+t} = 1 - \exp\{-\int_0^i v(k_0+t+h)dh\} \approx v(k_0+t)i + O(i)$$

$$\frac{i\mathcal{X}_{k_0+t}}{1-i\mathcal{X}_{k_0+t}} = \frac{1-\exp\{-\int_0^i v(\vartheta_0+t+h)dh\}}{\exp\{-\int_0^i v(\vartheta_0+t+h)dh\}} = \exp\{\int_0^i v(k_0+t+hdh\} - 1 \approx vk0+ti+O(i)$$

As $i \to 0$, $\frac{i\mathcal{X}_{k_0+t}}{1-i\mathcal{X}_{k_0+t}} = v(k_0+t)dt$, $i\mathcal{X}_{k_0+t} = v(k_0+t)dt bi \to bdt$, $\frac{\mathcal{R}_{t+i}(t)-\mathcal{R}_t}{\mathcal{R}_t} \to \frac{d\mathcal{R}_t(t)}{\mathcal{R}_t(t)}$, $\frac{\mathcal{S}_{t+i}(t)-\mathcal{S}_t}{\mathcal{S}_t} \to \frac{d\mathcal{S}_t(t)}{\mathcal{S}_t(t)}$, $\frac{\mathcal{L}_{t+i}(t)-\mathcal{L}_t}{\mathcal{L}_t} \to \frac{d\mathcal{L}_t(t)}{\mathcal{L}_t(t)}$ (1.7)

From (1.1), (1.2), (1.3) and (1.7), (1.6) becomes

$$d\mathcal{U}(t) = \begin{pmatrix} \left\{ \begin{matrix} \mathcal{U}(t)\begin{pmatrix} \mu_2\left(\Delta_1 \mathcal{Z}(t) + \frac{1}{k-k_0-t}\right) \\ +\mu_3(\Delta_2 + \frac{1}{k-k_0-t}) + r_1 \\ (1-n)\frac{1}{k-k_0-t} \end{pmatrix} \\ +b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right) \end{matrix} \right\} dt + \\ \mathcal{U}(t)\begin{pmatrix} \mu_3(n_2 d\mathcal{W}_l + n_3 d\mathcal{W}_m) \\ +\mu_2\sqrt{\mathcal{Z}(t)}d\mathcal{W}_s \end{pmatrix} \end{pmatrix} \mathcal{U}(0) = u_0 \quad (1.8)$$

Where k is the maximal age of the life table and $v(t)$is the force function given by

$$v(t) = \frac{1}{k-t}, \qquad 0 \le t < k$$

2. Mean-Variance Utility and Extended HJB equation

Consider a pension fund manager whose interest is to maximize his profit while penalising risk by using the mean-varianceutility function given as

$$\mathcal{H}(t,u,z) = \sup_{\mu}\left\{E_{t,u,z}\mathcal{U}^{\mu}(T) - Var_{t,u,z}\mathcal{U}^{\mu}(T)\right\} \tag{2.1}$$

Applying the game theoretic method described in Björk and Murgoci (2010) the mean-variance control problem in (2.1) is similar to the following Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{H}(t,u,z)$.

$$\begin{cases} \mathcal{J}(t,u,z,\mu) = E_{t,u,z}[\mathcal{U}^{\mu}(T)] - \frac{\gamma}{2}Var_{t,u,z}[\mathcal{U}^{\mu}(T)] \\ \mathcal{J}(t,u,z,\mu) \ = E_{t,u,z}[\mathcal{U}^{\mu}(T)] - \frac{\gamma}{2}\left(E_{t,u,z}[\mathcal{U}^{\mu}(T)^2] - (E_{t,u,z}[\mathcal{U}^{\mu}(T)])^2\right) \\ \mathcal{H}(t,u,z) \ = \sup\limits_{\mu} \mathcal{J}(t,u,z,\mu) \end{cases}$$

From Björk and Murgoci(2010) the optimal portfolio policy μ^* satisfies:

$$\mathcal{H}(t,u,z) = \sup_{\mu} \mathcal{J}(t,u,z,\mu^*)$$

$\gamma$is a constant representing risk aversion coefficient of the members.

Let $p^{\mu}(t,u,z) = E_{t,u,z}[\mathcal{U}^{\mu}(T)]$, $q^{\mu}(t,u,z) = E_{t,u,z}[\mathcal{U}^{\mu}(T)^2]$ then

$$\mathcal{H}(t,u,z) \quad = \sup_{\mu} e\big(t,u,z,p^{\mu}(t,u,z),q^{\mu}(t,u,z)\big)$$

Where,

$$e(t,u,z,p,q) = p - \frac{\gamma}{2}(q - p^2) \tag{2.2}$$

Theorem 4.5.1 (verification theorem). If there exist three real functions $\mathcal{E},\mathcal{F},\mathcal{G} : [0,T] \times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\begin{cases} \sup\limits_{\mu}\left\{\begin{array}{c} \mathcal{E}_t - e_t \\ + \left[u\left(\begin{array}{c} r_1 + (1-n)\frac{1}{k-k_0-t} + \mu_2\left(\Delta_1 \mathcal{Z}(\mathrm{t}) + \frac{1}{k-k_0-t}\right) \\ +\mu_3(\Delta_2 + \frac{1}{k-k_0-t}) + b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\end{array}\right) + b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\right](\mathcal{E}_u - e_u) \\ +\hbar(\nabla - z)(\mathcal{E}_z - e_z) \\ +\frac{1}{2}u^2[\mu_2^2 z + \mu_3^2(n_2^2+n_3^2)](\mathcal{E}_{uu} - \mathcal{A}_{uu}) \\ +\frac{1}{2}n_1^2 z(\mathcal{E}_{zz} - \mathcal{A}_{zz}) + (u\varepsilon z n_1\mu_2)(\mathcal{E}_{uz} - \mathcal{A}_{uz}) \end{array}\right\} = 0 \\ \mathcal{E}(T,u,z) = \ e(t,u,z,u^2) \end{cases} \tag{2.3}$$

Where,

$$\begin{cases} \mathcal{A}_{uu} = e_{uu} + 2e_{up}p_u + 2e_{uq}q_u + e_{pp}p_u^2 + 2e_{pq}p_u q_u + e_{qq}q_u^2 = \gamma\mathcal{F}_u^2 \\ \mathcal{A}_{zz} = \gamma\mathcal{F}_z^2, \mathcal{A}_{zu} = \gamma\mathcal{F}_u\mathcal{F}_z \end{cases} \tag{2.4}$$

$$\begin{cases} \left.\begin{array}{c} \mathcal{F}_t \\ + \left[u\left(\begin{array}{c} r_1 + (1-n)\frac{1}{k-k_0-t} + \mu_2\left(\Delta_1 \mathcal{Z}(\mathrm{t}) + \frac{1}{k-k_0-t}\right) \\ +\mu_3(\Delta_2 + \frac{1}{k-k_0-t}) + b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\end{array}\right) + b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\right]\mathcal{F}_u \\ +\hbar(\nabla - z)\mathcal{F}_z \\ +\frac{1}{2}u^2[\mu_2^2 z + \mu_3^2(n_2^2+n_3^2)]\mathcal{F}_{uu} \\ +\frac{1}{2}n_1^2 z\mathcal{F}_{zz} + (u\varepsilon z n_1\mu_2)\mathcal{F}_{uz} \end{array}\right\} = 0 \\ \mathcal{F}(T,u,z) = \ u \end{cases} \tag{2.5}$$

$$\left\{\begin{array}{c} \mathcal{G}_t \\ +\left[u\left(\begin{array}{c} \mathscr{r}_1+(1-n)\frac{1}{k-k_0-t}+\mu_2\left(\Delta_1 \mathcal{Z}(\mathrm{t})+\frac{1}{k-k_0-t}\right) \\ +\mu_3(\Delta_2+\frac{1}{k-k_0-t})+b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right) \end{array}\right)+b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\right]\mathcal{G}_u \\ +\hbar(\nabla-z)\mathcal{G}_z \\ +\frac{1}{2}u^2[\mu_2^2 z+\mu_3^2(n_2^2+n_3^2)]\mathcal{G}_{uu} \\ +\frac{1}{2}n_1^2 z\mathcal{G}_{zz}+(u\varepsilon z n_1\mu_2)\mathcal{G}_{uz} \end{array}\right\}=0 \quad (2.6)$$

$$\mathcal{G}(T,u,z)=u^2$$

Then $\mathcal{H}(t,u,z)=\mathcal{E}(t,u,z), p^{\mu^*}=\mathcal{F}(t,u,z), q^{\mu^*}=\mathcal{G}(t,u,z)$for the optimal control lawsμ^*.

Proof:

The details of the proof can be found in (He and Liang, 2009; Liang and Huang, 2011; Li and Zeng, 2011)

3. Optimal Control Laws and Efficient Frontier

In this section, we attempt to solve the extended HJB equation in (2.3) and (2.5) for the optimal control laws of the three assets and also the efficient frontier.

Proposition 3.1

The optimal control laws for the three assets are given as

$$\mu_1{}^*=1-\left[\begin{array}{c} \frac{\Delta_1 e^{\mathscr{r}_1(t-T)}}{\gamma u(\hbar+\varepsilon n_1\Delta_1)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{1-n}\left\{\hbar+\varepsilon n_1\Delta_1 e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}+\frac{1}{z}\left(\frac{1}{k-k_0-t}\right)\right\} \\ +\frac{e^{\mathscr{r}_1(t-T)}}{\gamma u(n_2^2+n_3^2)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{1-n}\left(\Delta_2+\left(\frac{1}{k-k_0-t}\right)\right) \end{array}\right] \quad (3.1)$$

$$\mu_2{}^*=\frac{\Delta_1 e^{\mathscr{r}_1(t-T)}}{\gamma u(\hbar+\varepsilon n_1\Delta_1)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{1-n}\left\{\hbar+\varepsilon n_1\Delta_1 e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}+\frac{1}{z}\left(\frac{1}{k-k_0-t}\right)\right\} \quad (3.2)$$

$$\mu_3{}^*=\frac{e^{\mathscr{r}_1(t-T)}}{\gamma u(n_2^2+n_3^2)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{1-n}\left(\Delta_2+\left(\frac{1}{k-k_0-t}\right)\right) \quad (3.3)$$

Proof

From (2.2),

$$e_t=e_u=e_z=e_{uu}=e_{zz}=e_{uz}=e_{pz}=e_{qz}=e_{up}=e_{uq}=e_{pq}=e_{qq}=0,\ e_p=1+\gamma p,$$

$$e_{pp}=\gamma,\ e_q=-\frac{\gamma}{2} \quad (3.4)$$

Substituting (3.4) into (2.3) and differentiating it with respect to $\mu_2$and μ_3, and solving for μ_2 and μ_3, we have:

$$\mu_2{}^*=-\left[\frac{(\Delta_1 z+\frac{1}{k-k_0-t})e_u+(\mathcal{E}_{uz}-\gamma\mathcal{F}_u\mathcal{F}_z)\varepsilon n_1}{zu(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\right] \quad (3.5)$$

$$\mu_3{}^*=-\left[\frac{(\Delta_2+\frac{1}{k-k_0-t})e_u}{u\left((\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)\right)(n_2^2+n_3^2)}\right] \quad (3.6)$$

Substituting (4.5) and (4.6) into (3.3) and (3.5) we have

$$\mathcal{E}_t+\left[\left(\mathscr{r}_1+(1-n)\frac{1}{k-k_0-t}\right)u+b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\right]\mathcal{E}_u+\hbar(\nabla-z)\mathcal{E}_z+\frac{1}{2}(\mathcal{E}_{zz}-\gamma\mathcal{F}_z^2)n_1^2 z$$
$$-\frac{1}{2}\frac{\mathcal{E}_u^2}{(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\left(\frac{\left(\Delta_1 z+\frac{1}{k-k_0-t}\right)^2}{z}+\frac{\left(\Delta_2+\frac{1}{k-k_0-t}\right)^2}{(n_2^2+n_3^2)}\right)-\frac{1}{2}\frac{(\mathcal{E}_{uz}-\gamma\mathcal{F}_u\mathcal{F}_z)^2}{(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\varepsilon^2 n_1^2=0 \quad (3.7)$$

$$
\begin{aligned}
&\mathcal{F}_t\\
&+\left[\left(r_1+(1-n)\frac{1}{k-k_0-t}\right)u+b\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)\right]\mathcal{F}_u+\hbar(\nabla-z)\mathcal{F}_z\\
&+\frac{1}{2}n_1^2z\mathcal{F}_{zz}-\frac{\mathcal{E}_u\mathcal{F}_u}{(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\left(\frac{\left(\Delta_1 z+\frac{1}{k-k_0-t}\right)^2}{z}+\frac{\left(\Delta_2+\frac{1}{k-k_0-t}\right)^2}{(n_2^2+n_3^2)}\right)\\
&-\frac{\mathcal{F}_u(\mathcal{E}_{uz}-\gamma\mathcal{F}_u\mathcal{F}_z)}{(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\left(\Delta_1 z+\frac{1}{k-k_0-t}\right)\varepsilon n_1\\
&+\frac{1}{2}\left(\begin{array}{c} z\left(\frac{\left(\Delta_1 z+\frac{1}{k-k_0-t}\right)e_u+(\mathcal{E}_{uz}-\gamma\mathcal{F}_u\mathcal{F}_z)\varepsilon n_1}{z(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\right)^2\\ +(n_2^2+n_3^2)\left(\frac{(\Delta_2+\frac{1}{k-k_0-t})e_u}{(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)(n_2^2+n_3^2)}\right)^2\end{array}\right)\mathcal{F}_{zu}\\
&+\varepsilon n_1\left(\frac{\left(\Delta_1 z+\frac{1}{k-k_0-t}\right)e_u+(\mathcal{E}_{uz}-\gamma\mathcal{F}_u\mathcal{F}_z)\varepsilon n_1}{(\mathcal{E}_{uu}-\gamma\mathcal{F}_u^2)}\right)\mathcal{F}_u=0
\end{aligned}
\tag{3.8}
$$

Next, we assume a solution for $\mathcal{E}(t,u,z)$ and $\mathcal{F}(t,u,z)$ as follows:

$$
\begin{cases}
\mathcal{E}(t,u,z)=u\mathcal{B}_1(t)+\frac{z}{\gamma}\mathcal{B}_2(t)+\frac{1}{\gamma}\mathcal{B}_3(t),\ \mathcal{B}_1(T)=1,\ \mathcal{B}_2(T)=0,\ \mathcal{B}_3(T)=0\\
\mathcal{F}(t,u,z)=u\mathcal{C}_1(t)+\frac{z}{\gamma}\mathcal{C}_2(t)+\frac{1}{\gamma}\mathcal{C}_3(t),\mathcal{C}_1(T)=1,\ \mathcal{C}_2(T)=0,\ \mathcal{C}_3(T)=0\\
\mathcal{E}_t=u\frac{d\mathcal{B}_1(t)}{dt}+\frac{z}{\gamma}\frac{d\mathcal{B}_2(t)}{dt}+\frac{1}{\gamma}\frac{d\mathcal{B}_3(t)}{dt},\mathcal{E}_u=\mathcal{B}_1(t),\mathcal{E}_{uu}=0,\mathcal{E}_z=\frac{1}{\gamma}\mathcal{B}_2(t),\mathcal{E}_{zz}=0\\
\mathcal{F}_t=u\frac{d\mathcal{C}_1(t)}{dt}+\frac{z}{\gamma}\frac{d\mathcal{C}_2(t)}{dt}+\frac{1}{\gamma}\frac{d\mathcal{C}_3(t)}{dt},\mathcal{F}_u=\mathcal{C}_1(t),\mathcal{F}_{uu}=0,\mathcal{F}_z=\frac{1}{\gamma}\mathcal{C}_2(t),\mathcal{F}_{zz}=0
\end{cases}
\tag{3.9}
$$

Substituting (4.9) into (4.7) and (4.8), we have

$$
\begin{cases}
\frac{d\mathcal{B}_1(t)}{dt}+\left(r_1+(1-n)\frac{1}{k-k_0-t}\right)\mathcal{B}_1(t)=0\quad,\mathcal{B}_1(T)=1\\
\frac{d\mathcal{B}_2(t)}{dt}-\hbar\mathcal{B}_2+\frac{(\varepsilon^2-1)n_1^2{\mathcal{C}_2}^2}{2}+\frac{\Delta_1^2{\mathcal{B}_1}^2}{2{\mathcal{C}_1}^2}-\frac{\varepsilon n_1\Delta_1\mathcal{B}_1\mathcal{C}_2}{\mathcal{C}_1}=0\ ,\ \mathcal{B}_2(T)=0\\
\frac{d\mathcal{B}_3(t)}{dt}+\hbar\nabla\mathcal{B}_2+\mathcal{B}_1 b\gamma\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)+\frac{\left(\Delta_2+\frac{1}{k-k_0-t}\right)^2\mathcal{B}_1\mathcal{C}_1}{(n_2^2+n_3^2)\mathcal{C}_1^2}\\
-\left(\frac{1}{k-k_0-t}\right)\varepsilon n_1\Delta_1\mathcal{C}_1+\left(\frac{1}{k-k_0-t}\right)\Delta_1\frac{{\mathcal{B}_1}^2}{{\mathcal{C}_1}^2}+\frac{1}{2}\frac{\left(\frac{1}{k-k_0-t}\right)^2\mathcal{B}_1^2}{z\mathcal{C}_1^2}=0,\qquad\mathcal{B}_3(T)=0
\end{cases}
\tag{3.10}
$$

$$
\begin{cases}
\frac{d\mathcal{C}_1(t)}{dt}+\left(r_1+(1-n)\frac{1}{k-k_0-t}\right)\mathcal{C}_1(t)=0\qquad,\mathcal{C}_1(T)=1\\
\frac{d\mathcal{C}_2(t)}{dt}-\hbar\mathcal{C}_2+\frac{\Delta_1^2\mathcal{B}_1}{\mathcal{C}_1}-\frac{\varepsilon n_1\Delta_1\mathcal{B}_1\mathcal{C}_2}{\mathcal{C}_1}=0,\qquad\mathcal{C}_2(T)=0\\
\frac{d\mathcal{C}_3(t)}{dt}+\hbar\nabla\mathcal{C}_2+\mathcal{C}_1 b\gamma\left(\frac{k-k_0-(1+n)t}{k-k_0-t}\right)+\frac{\left(\Delta_2+\frac{1}{k-k_0-t}\right)^2\mathcal{B}_1\mathcal{C}_1}{2(n_2^2+n_3^2)\mathcal{C}_1^2}\\
-\left(\frac{1}{k-k_0-t}\right)\varepsilon n_1\Delta_1\mathcal{C}_2+2\left(\frac{1}{k-k_0-t}\right)\Delta_1\frac{\mathcal{B}_1\mathcal{C}_1}{{\mathcal{C}_1}^2}+\frac{1}{2}\frac{\left(\frac{1}{k-k_0-t}\right)^2\mathcal{B}_1\mathcal{C}_1}{z\mathcal{C}_1^2}=0,\qquad\mathcal{C}_3(T)=0
\end{cases}
\tag{3.11}
$$

Solving (3.10) and (3.11), we have

$$\mathcal{B}_1(t)=e^{r_1(T-t)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{n-1}$$

$$\mathcal{C}_1(t)=e^{r_1(T-t)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{n-1}$$

$$\mathcal{B}_2(t) = \begin{pmatrix} \dfrac{\varepsilon n_1 \Delta_1^3}{\hbar + \varepsilon n_1 \Delta_1} \begin{Bmatrix} \dfrac{1}{\hbar}\left(e^{\hbar(t-T)} - 1\right) \\ + \dfrac{1}{\varepsilon n_1 \Delta_1}\left(e^{\hbar(t-T)} - e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right) \end{Bmatrix} \\ + \dfrac{\Delta_1^2}{2\hbar}\left(1 - e^{\hbar(t-T)}\right) + \dfrac{n_1^2 \Delta_1^3 (\varepsilon^2 - 1)}{2(\hbar + \varepsilon n_1 \Delta_1)^2} \begin{Bmatrix} \dfrac{1}{\hbar}\left(e^{\hbar(t-T)} - 1\right) \\ + \dfrac{2e^{\hbar(t-T)}}{\varepsilon \sigma_1 k_1}\left(1 - e^{\varepsilon n_1 \Delta_1 (t-T)}\right) + \\ \dfrac{e^{\hbar(t-T)}}{\hbar + 2\varepsilon n_1 \Delta_1}\left(e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)} - 1\right) \end{Bmatrix} \end{pmatrix}$$

$$\mathcal{C}_2(t) = \frac{\Delta_1^2}{\hbar + \varepsilon n_1 \Delta_1}\left(1 - e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)$$

$$\mathcal{B}_3(t) = \begin{pmatrix} -\hbar\nabla \displaystyle\int_t^T \mathcal{B}_2(\tau)\, d\tau + b\gamma \int_t^T \mathcal{C}_1(\tau)\left(\frac{k - k_0 - (1+n)\tau}{k - k_0 - \tau}\right) d\tau \\ + \begin{pmatrix} \Delta_1 \\ + \dfrac{\Delta_2^2 + 2\Delta_2}{2(n_2^2 + n_3^2)} \\ + \dfrac{\left(\frac{1}{(n_2^2+n_3^2)} + \frac{1}{z}\right)}{2(k - k_0 - t)(k - k_0 - T)} \end{pmatrix}(T - t) \end{pmatrix}$$

$$\mathcal{C}_3(t) = \begin{pmatrix} -\hbar\nabla \displaystyle\int_t^T \mathcal{C}_2(\tau)\, d\tau + b\gamma \int_t^T \mathcal{C}_1(\tau)\left(\frac{k - k_0 - (1+n)\tau}{k - k_0 - \tau}\right) d\tau \\ + \begin{pmatrix} \dfrac{\Delta_2^2 + 2\Delta_2}{2(n_2^2 + n_3^2)} \\ + \dfrac{\left(\frac{1}{(n_2^2+n_3^2)} + \frac{1}{z}\right)}{2(k - k_0 -)(k - k_0 - T)} \end{pmatrix}(T - t) + \left(2\Delta_1 - \dfrac{\Delta_1^2}{\hbar + \varepsilon n_1 \Delta_1}\right) ln\left(\dfrac{k - k_0 - T}{k - k_0 - t}\right) \end{pmatrix}$$

$$\mathcal{E}(t,u,z) = \begin{pmatrix} u e^{r_1(T-t)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{n-1} \\ + \frac{z}{\gamma}\begin{pmatrix} \frac{\varepsilon n_1 \Delta_1^3}{\hbar + \varepsilon n_1 \Delta_1} \begin{Bmatrix} \frac{1}{\hbar}\left(e^{\hbar(t-T)} - 1\right) \\ + \frac{1}{\varepsilon n_1 \Delta_1}\left(e^{\hbar(t-T)} - e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right) \end{Bmatrix} \\ + \frac{\Delta_1^2}{2\hbar}\left(1 - e^{\hbar(t-T)}\right) + \frac{n_1^2 \Delta_1^3 (\varepsilon^2 - 1)}{2(\hbar + \varepsilon n_1 \Delta_1)^2} \begin{Bmatrix} \frac{1}{\hbar}\left(e^{\hbar(t-T)} - 1\right) \\ + \frac{2e^{\hbar(t-T)}}{\varepsilon \sigma_1 k_1}\left(1 - e^{\varepsilon n_1 \Delta_1 (t-T)}\right) + \\ \frac{e^{\hbar(t-T)}}{\hbar + 2\varepsilon n_1 \Delta_1}\left(e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)} - 1\right) \end{Bmatrix} \end{pmatrix} \\ + \frac{1}{\gamma}\begin{pmatrix} -\hbar\nabla \int_t^T \mathcal{B}_2(\tau)\, d\tau + b\gamma \int_t^T \mathcal{C}_1(\tau)\left(\frac{k-k_0-(1+n)\tau}{k-k_0-\tau}\right) d\tau \\ + \begin{pmatrix} \Delta_1 \\ + \frac{\Delta_2^2 + 2\Delta_2}{2(n_2^2+n_3^2)} \\ + \frac{\left(\frac{1}{(n_2^2+n_3^2)} + \frac{1}{z}\right)}{2(k-k_0-t)(k-k_0-T)} \end{pmatrix}(T - t) \end{pmatrix} \end{pmatrix} \tag{3.12}$$

$$\mathcal{F}(t,u,z)=\left(\begin{array}{c} u e^{r_1(T-t)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{n-1} \\ +\frac{z}{\gamma}\left(\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\left(1-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\right) \\ +\frac{1}{\gamma}\left(\begin{array}{c} -\hbar\nabla\int_t^T \mathcal{C}_2(\tau)\,d\tau+b\gamma\int_t^T \mathcal{C}_1(\tau)\left(\frac{k-k_0-(1+n)\tau}{k-k_0-\tau}\right)d\tau \\ +\left(\begin{array}{c} \frac{\Delta_2^2+2\Delta_2}{2\left(n_2^2+n_3^2\right)} \\ +\frac{\left(\frac{1}{\left(n_2^2+n_3^2\right)}+\frac{1}{z}\right)}{2(k-k_0-t)(k-k_0-T)} \end{array}\right)(T-t)+\left(2\Delta_1-\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\right)ln\left(\frac{k-k_0-T}{k-k_0-t}\right) \end{array}\right) \end{array}\right) \tag{3.13}$$

Substituting $\mathcal{E}_u=\mathcal{B}_1(t)$,$\mathcal{F}_u=\mathcal{C}_1(t)$, $\mathcal{E}_{uu}=0$, and ,$\mathcal{F}_z=\frac{1}{\gamma}\mathcal{C}_2(t)$ into (3.5) and (3.6) we obtain (3.1), (3.2) and (3.3).

Proposition 3.2

The efficient frontier of the pension members is given by

$$E_{t,u,z}[\mathcal{U}^{\mu}(T)]=\left(\begin{array}{c} u e^{r_1(T-t)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{n-1}+b\int_t^T \mathcal{C}_1(\tau)\left(\frac{k-k_0-(1+n)\tau}{k-k_0-\tau}\right)d\tau \\ +\sqrt{\frac{Var_{t,u,z}[\mathcal{U}^{\mu}(T)]}{J(t)}}\left(\begin{array}{c} z\left(\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\left(1-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\right) \\ -\hbar\nabla\int_t^T \mathcal{C}_2(\tau)\,d\tau+\left(\begin{array}{c} \frac{\Delta_2^2+2\Delta_2}{2\left(n_2^2+n_3^2\right)} \\ +\frac{\left(\frac{1}{\left(n_2^2+n_3^2\right)}+\frac{1}{z}\right)}{2(k-k_0-)(k-k_0-T)} \end{array}\right)(T-t) \\ +\left(2\Delta_1-\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\right)ln\left(\frac{k-k_0-T}{k-k_0-t}\right) \end{array}\right) \end{array}\right) \tag{3.14}$$

Proof

$$Var_{t,u,z}[\mathcal{U}^{\mu^*}(T)]=E_{t,u,z}[\mathcal{U}^{\mu}(T)^2]-(E_{t,u,z}[\mathcal{U}^{\mu}(T)])^2=\frac{2}{\gamma}\left(\mathcal{F}(t,u,z)-\mathcal{E}(t,u,z)\right)$$

$$Var_{t,u,z}[\mathcal{U}^{\mu^*}(T)]=\frac{1}{\gamma^2}\left(\begin{array}{c} \left(\frac{2z\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\left(1-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\right) \\ -z\left(\begin{array}{c} \frac{2\varepsilon n_1\Delta_1^3}{\hbar+\varepsilon n_1\Delta_1}\left\{\begin{array}{c} \frac{1}{\hbar}\left(e^{\hbar(t-T)}-1\right) \\ +\frac{1}{\varepsilon n_1\Delta_1}\left(e^{\hbar(t-T)}-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right) \end{array}\right\} \\ +\frac{\Delta_1^2}{\hbar}\left(1-e^{\hbar(t-T)}\right) \\ +\frac{n_1^2\Delta_1^3(\varepsilon^2-1)}{(\hbar+\varepsilon n_1\Delta_1)^2}\left\{\begin{array}{c} \frac{1}{\hbar}\left(e^{\hbar(t-T)}-1\right) \\ +\frac{2e^{\hbar(t-T)}}{\varepsilon\sigma_1 k_1}\left(1-e^{\varepsilon n_1\Delta_1(t-T)}\right)+ \\ \frac{e^{\hbar(t-T)}}{\hbar+2\varepsilon n_1\Delta_1}\left(e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}-1\right) \end{array}\right\} \end{array}\right) \\ +\left(2\hbar\nabla\left(\int_t^T \mathcal{B}_2(\tau)\,d\tau-\int_t^T \mathcal{C}_2(\tau)\,d\tau\right)+\left(2\Delta_1-\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\right)ln\left(\frac{k-k_0-T}{k-k_0-t}\right)^2-2\Delta_1(T-t)\right) \end{array}\right) \tag{3.15}$$

$$Var_{t,u,z}[\mathcal{U}^{\mu^*}(T)]=\frac{1}{\gamma^2}J(t)$$

Where

$$
\mathcal{T}(t)=\left(\begin{array}{c}
\left(\dfrac{2z\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\left(1-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\right)\\
-z\left(\begin{array}{c}
\dfrac{2\varepsilon n_1\Delta_1^3}{\hbar+\varepsilon n_1\Delta_1}\left\{\begin{array}{c}\dfrac{1}{\hbar}\left(e^{\hbar(t-T)}-1\right)\\ +\dfrac{1}{\varepsilon n_1\Delta_1}\left(e^{\hbar(t-T)}-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\end{array}\right\}\\
+\dfrac{\Delta_1^2}{\hbar}\left(1-e^{\hbar(t-T)}\right)\\
+\dfrac{n_1^2\Delta_1^3(\varepsilon^2-1)}{(\hbar+\varepsilon n_1\Delta_1)^2}\left\{\begin{array}{c}\dfrac{1}{\hbar}\left(e^{\hbar(t-T)}-1\right)\\ +\dfrac{2e^{\hbar(t-T)}}{\varepsilon\sigma_1 k_1}\left(1-e^{\varepsilon n_1\Delta_1(t-T)}\right)+\\ \dfrac{e^{\hbar(t-T)}}{\hbar+2\varepsilon n_1\Delta_1}\left(e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}-1\right)\end{array}\right\}
\end{array}\right)\\
+\left(2\hbar\nabla\left(\int_t^T\mathcal{B}_2(\tau)\,d\tau-\int_t^T\mathcal{C}_2(\tau)\,d\tau\right)+\left(2\Delta_1-\dfrac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\right)ln\left(\dfrac{k-k_0-T}{k-k_0-t}\right)^2-2\Delta_1(T-t)\right)
\end{array}\right)
$$

$$
\frac{1}{\gamma}=\frac{\sqrt{Var_{t,u,z}[\mathcal{U}^{\mu^*}(T)}}{\sqrt{\mathcal{T}(t)}} \tag{3.16}
$$

$$
E_{t,u,z}[Z^{\mu^*}(T)]=\left(\begin{array}{c}
ue^{r_1(T-t)}\left(\frac{k-k_0-T}{k-k_0-t}\right)^{n-1}\\
+\frac{z}{\gamma}\left(\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\left(1-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\right)\\
+\frac{1}{\gamma}\left(\begin{array}{c}
-\hbar\nabla\int_t^T\mathcal{C}_2(\tau)\,d\tau+b\gamma\int_t^T\mathcal{C}_1(\tau)\left(\frac{k-k_0-(1+n)\tau}{k-k_0-\tau}\right)d\tau\\
+\left(\begin{array}{c}\frac{\Delta_2^2+2\Delta_2}{2(n_2^2+n_3^2)}\\ +\frac{\left(\frac{1}{(n_2^2+n_3^2)}+\frac{1}{z}\right)}{2(k-k_0-t)(k-k_0-T)}\end{array}\right)(T-t)+\left(2\Delta_1-\frac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\right)ln\left(\frac{k-k_0-T}{k-k_0-t}\right)
\end{array}\right)
\end{array}\right) \tag{3.17}
$$

Substituting (3.16) in (3.17), we have

$$
E_{t,u,z}[\mathcal{U}^{\mu}(T)]=\left(\begin{array}{c}
ue^{r_1(T-t)}\left(\dfrac{k-k_0-T}{k-k_0-t}\right)^{n-1}+b\displaystyle\int_t^T\mathcal{C}_1(\tau)\left(\dfrac{k-k_0-(1+n)\tau}{k-k_0-\tau}\right)d\tau\\
+\sqrt{\dfrac{Var_{t,u,z}[\mathcal{U}^{\mu}(T)]}{\mathcal{T}(t)}}\left(\begin{array}{c}
z\left(\dfrac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\left(1-e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)}\right)\right)\\
-\hbar\nabla\displaystyle\int_t^T\mathcal{C}_2(\tau)\,d\tau+\left(\begin{array}{c}\dfrac{\Delta_2^2+2\Delta_2}{2(n_2^2+n_3^2)}\\ +\dfrac{\left(\frac{1}{(n_2^2+n_3^2)}+\frac{1}{z}\right)}{2(k-k_0-)(k-k_0-T)}\end{array}\right)(T-t)\\
+\left(2\Delta_1-\dfrac{\Delta_1^2}{\hbar+\varepsilon n_1\Delta_1}\right)ln\left(\dfrac{k-k_0-T}{k-k_0-t}\right)
\end{array}\right)
\end{array}\right)
$$

Remark 3.1: If $n=1$, the optimal control laws for the three assets in (3.1), (3.2) and (3.3) reduces to a case when there is return of contributions as follows

$$\mu_1{}^* = 1 - \left[\begin{array}{c}\frac{\Delta_1 e^{r_1(t-T)}}{\gamma u\,(\hbar+\varepsilon n_1\Delta_1)}\left\{\hbar + \varepsilon n_1\Delta_1 e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)} + \frac{1}{z}\left(\frac{1}{k-k_0-t}\right)\right\} \\ + \frac{e^{r_1(t-T)}}{\gamma u\,(n_2^2+n_3^2)}\left(\Delta_2 + \left(\frac{1}{k-k_0-t}\right)\right)\end{array}\right] \quad (3.18)$$

$$\mu_2{}^* = \frac{\Delta_1 e^{r_1(t-T)}}{\gamma u\,(\hbar+\varepsilon n_1\Delta_1)}\left\{\hbar + \varepsilon n_1\Delta_1 e^{(\hbar+\varepsilon n_1\Delta_1)(t-T)} + \frac{1}{z}\left(\frac{1}{k-k_0-t}\right)\right\} \quad (3.19)$$

$$\mu_3{}^* = \frac{e^{r_1(t-T)}}{\gamma u\,(n_2^2+n_3^2)}\left(\Delta_2 + \left(\frac{1}{k-k_0-t}\right)\right) \quad (3.20)$$

4. Numerical Simulations

In this section, we present some numerical results of the optimal control law with return of premium using the following parameters unless otherwise stated γ = 0.5,r_1 = 0.03,$\hbar$ =0.5,ε = 0.3, $n_1 = 0.1, n_2 = 0.3, n_3 = 0.5, \Delta_1$ = 0.4,Δ_2 = 0.5, $z_0 = 0.1$ u_0= 0.1,T = 40, t = 0:5:25,k = 100, k_0 = 20.

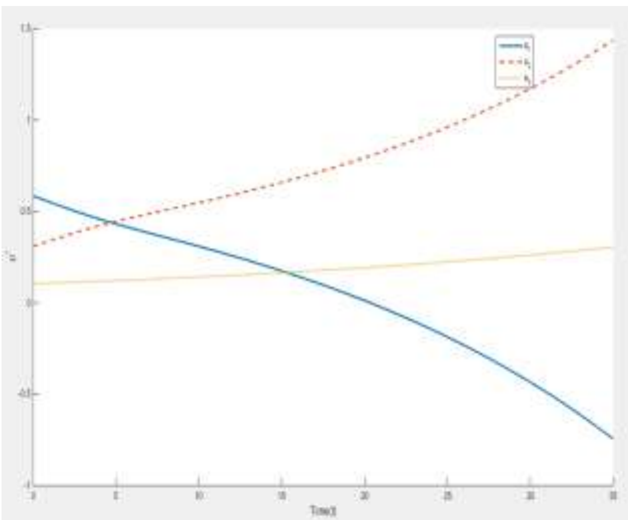
Fig. 1. Time evolution of $\mu_1{}^*$, $\mu_2{}^*$, and $\mu_3{}^*$

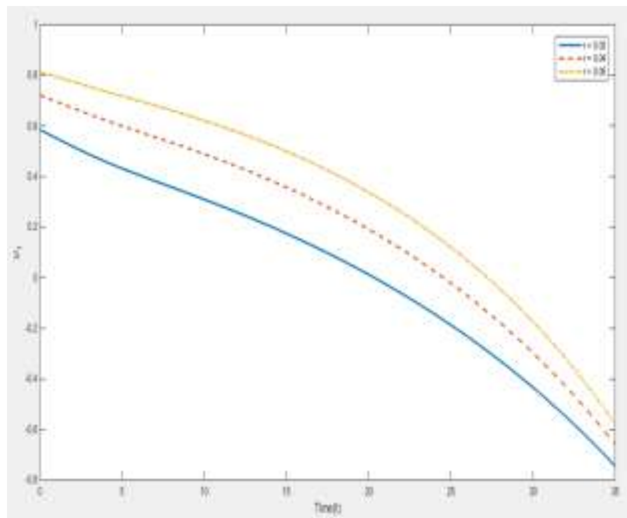
Fig. 3. Time evolution of $\mu_1{}^*$ with different r_1

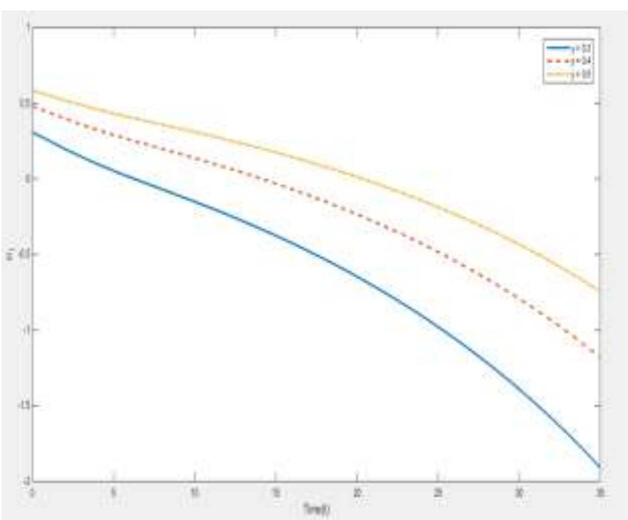
Fig. 2. Time evolution of $\mu_1{}^*$ with different γ.

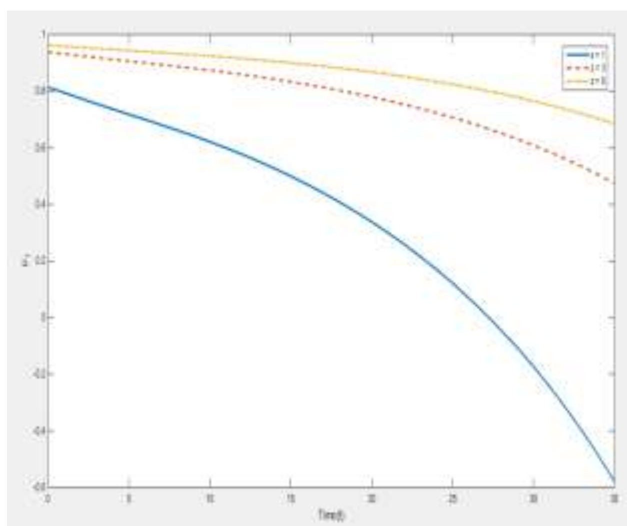
Fig. 4. Time evolution of $\mu_1{}^*$ with different u.

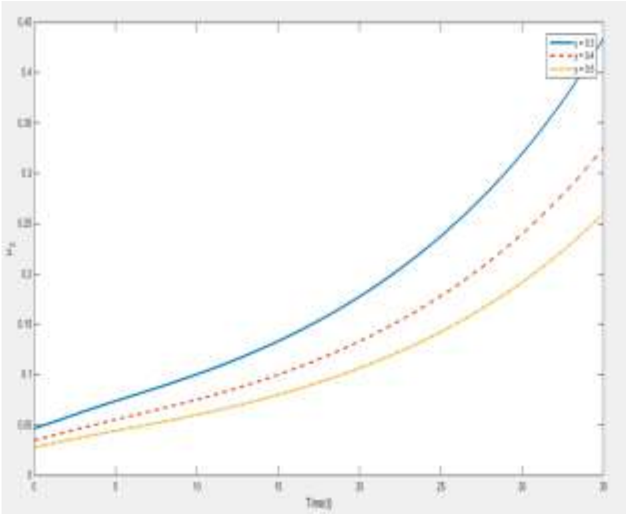

Fig. 5. Time evolution of the $\mu_2{}^*$ different γ.

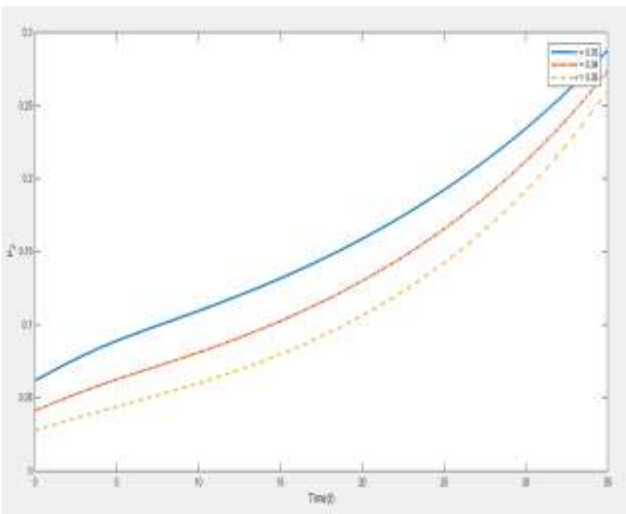

Fig. 6. Time evolution of $\mu_2{}^*$ with different r_1.

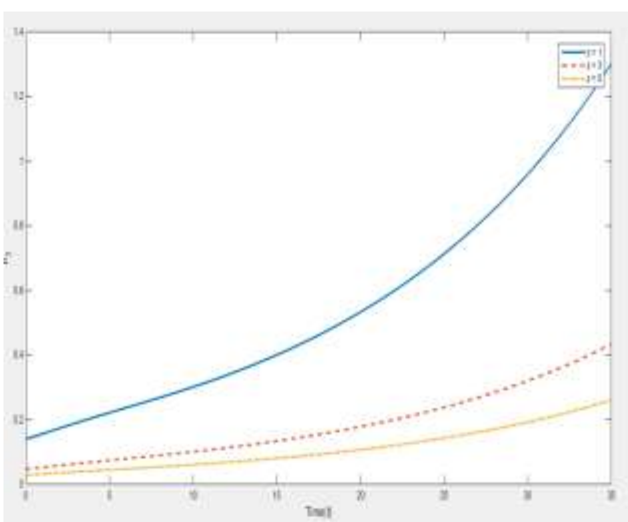

Fig. 7. Time evolution of $\mu_2{}^*$ with different u.

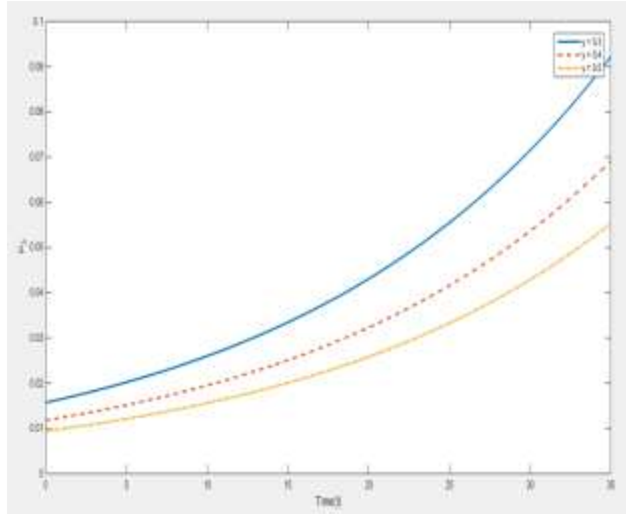

Fig. 8. Time evolution of $\mu_3{}^*$ with different γ.

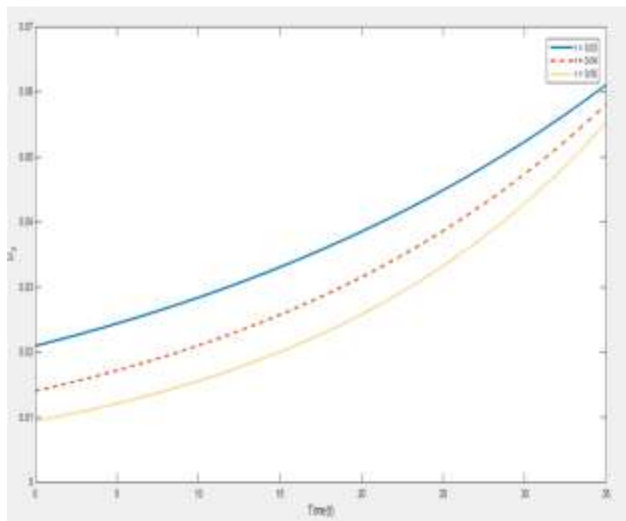

Fig. 9. Time evolution of $\mu_3{}^*$ with different r_1.

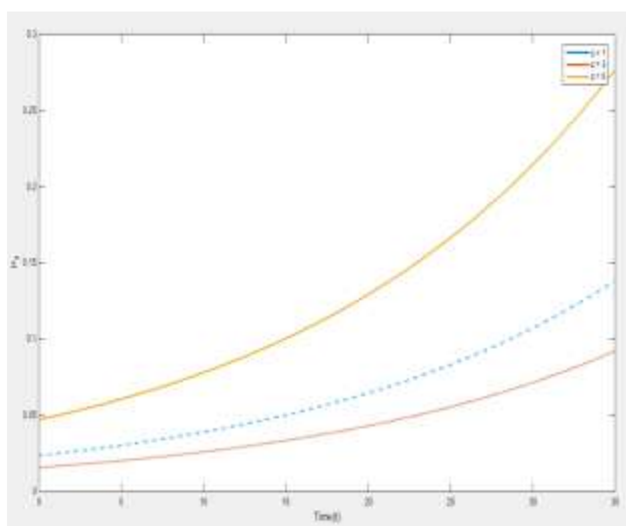

Fig. 10. Time evolution of $\mu_3{}^*$ with different u.

DISCUSSION

In Figure 1, we observed that the optimal control law for risk free asset decreases continuously with time while that of stock and loan increases continuously with time. From Figure 2, 3 and 4, the optimal control law of the risk free asset is directly proportional to the risk aversion coefficient of the pension members, initial wealth and the predetermined interest rate of the risk free asset while in Figures 5 and 8, we observed that the optimal control law of the risky assets are inversely proportional to risk aversion coefficient of the pension members; this implies that an investor with high risk aversion coefficient will invest less in equity and loan and vice versa. Figure 6 and 9 shows that the optimal control laws of the risky assets are inversely proportional to predetermined interest; the implication here is that, as the interest rate of the risk free asset increases, there is a decrease in investments in equity and loan. In a similar fashion, Figure 7 and 10 shows that the optimal control laws for the risky assets are inversely proportional to the initial wealth the implication is that when there sufficiently large amount in the pension purse, at the early stages of the investment, the fund managers will like to undertake lesser risk when compared to cases where initial wealth is small.

CONCLUSION

This work merged together, the study of optimal control law for a pension plan with and without return clause under Heston volatility model. We presented an investment model which takes into considerations members' monthly contributions, return accumulations with risk free interest to dead members' families for the case with return clause and investment in three different assets. The game theoretic approach was used to establish our optimization problem from the extended Hamilton Jacobi Bellman (HJB). The variable separation technique was used to obtain the optimal control laws for the three assets and the efficient frontier by solving the optimization problem. Numerical simulations were presented to demonstrate the effects of some parameters on the optimal control laws with observations that are presented with observations that the optimal control law for risk free asset decreases continuously with time while that of stock and loan increases continuously with time.

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